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ON THE BIRATIONAL GEOMETRY OF VARIETIES OF MAXIMAL ALBANESE DIMENSION

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1. Introduction

Combining the generic vanishing theorems of Green and Lazarsfeld [GL1], [GL2], the theory of Fourier Mukai transforms [Mu] and the results of Kollàr on higher direct images of dualizing sheaves [Ko1], [Ko2], it is possible to obtain surprisingly precise information about the birational geometry of varieties of maximal Albanese dimension. In [CH1] and [CH3] for example, it is shown that the following conjecture of Kollàr holds (cf. [Ko3])

Theorem. *If X is a smooth complex algebraic variety with $P_2(X) = 1$ and $q(X) = \dim(X)$, then X is birational to an abelian variety.*

In this paper we show how these techniques can be used to give effective criteria for a morphism of varieties of maximal Albanese dimension to be birational. In particular we prove the following

Theorem 1. *Let $f: X \to Y$ be a generically finite morphism of smooth complex projective n-dimensional varieties of maximal Albanese dimension. If the induced maps*

$$H^i(Y, \omega_Y \otimes P) \to H^i(Y, f_*\omega_X \otimes P)$$

are isomorphisms for all $i \geq 0$ and all $P \in \mathrm{Pic}^0(Y)$, then $X \to Y$ is a birational morphism.

As an application of this, we prove a refinement of [Hac] Theorem 3.1.

Corollary 2. *Let $\nu: X \to A$ be a morphism from a smooth variety of dimension n to an abelian variety of dimension $n+1$ such that $h^0(X, \omega_X \otimes \nu^* P) = 1$ for all $\mathcal{O}_A \neq P \in \mathrm{Pic}^0(A)$ and the map $\nu^*: H^0(A, \Omega^n_A) \to H^0(X, \Omega^n_X)$ is an isomorphism. Then ν is birational onto its image $\bar{X}$ and $\bar{X}$ is a principal polarization.*

It is well known that if $f: X \to Y$ is a morphism of n-dimensional smooth projective varieties of general type such that $P_m(X) = P_m(Y)$ for $m \gg 0$ then f is birational. If Y is of maximal Albanese dimension, it suffices to verify the above condition for $m = 2$.

Theorem 3. *Let $f: X \to Y$ be a dominant morphism of smooth n-dimensional complex projective algebraic varieties, Y of general type*

and of maximal Albanese dimension. If $P_2(X) = P_2(Y)$, then f is a birational morphism.

Next we turn our attention to the Albanese morphism. By a theorem of Kawamata [Ka], if $\kappa(X) = 0$ then the Albanese map of X is surjective. In particular $\dim(X) \geq q(X)$. If furthermore $q(X) = \dim(X)$ then $\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$ is a birational morphism. In [Ko4], Kollàr proves effective versions of this result. We further refine [Ko4, Theorem 11.2]

Theorem 4. *Let X be a smooth projective variety. If $P_2(X) = 1$ or $0 < P_m(X) \leq 2m - 3$ for some $m \geq 3$, then $\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$ is surjective.*

As remarked above, the case $P_2(X) = 1$, $q(X) = \dim(X)$ was completely understood in [CH1] and [CH3]. We study the case $P_3(X) = 2$, $q(X) = \dim(X)$.

Theorem 5. *Let X be a smooth projective variety. Then $P_3(X) = 2$ and $q(X) = \dim(X)$ iff:*

a) *there is a surjective map $\phi\colon \mathrm{Alb}(X) \to E$, where E is a curve of genus 1;*
b) *$\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$ is birational to a smooth double cover of $\mathrm{Alb}(X)$ defined by $Spec(\mathcal{O}_{\mathrm{Alb}(X)} \oplus \phi^* L^{-1} \otimes P)$, where L is a line bundle of E of degree 1 and $P \in \mathrm{Pic}^0(X) \setminus \phi^*\mathrm{Pic}^0(E)$ is 2-torsion. The branch locus of the cover is the union of two distinct fibers of ϕ.*

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Notation and conventions. We work over the field of complex numbers. We identify Cartier divisors and line bundles on a smooth variety, and we use the additive and multiplicative notation interchangeably. If X is a smooth projective variety, we let K_X be a canonical divisor, so that $\omega_X = \mathcal{O}_X(K_X)$, and we denote by $\kappa(X)$ the Kodaira dimension, by $q(X) := h^1(\mathcal{O}_X)$ the *irregularity* and by $P_m(X) := h^0(\omega_X^m)$ the *$m-th$ plurigenus*. We denote by $\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$ the Albanese map and by $\mathrm{Pic}^\tau(X)$ the subgroup of torsion elements of $\mathrm{Pic}^0(X)$. For a $\mathbb{Q}-$divisor D we let $\lfloor D \rfloor$ be the integral part and $\{D\}$ the fractional part. Numerical equivalence is denoted by $\equiv$ and we write $D \prec E$ if $E - D$ is an effective divisor. If $f\colon X \to Y$ is a morphism, we write $K_{X/Y} := K_X - f^*K_Y$ and we often denote by $F_{X/Y}$ the general fiber of f. The rest of the notation is standard in algebraic geometry.

2. Preliminaries

Here we recall several results from the literature and we prove some technical statements that will be needed later.

2.1. **The Albanese map and the Iitaka fibration.** Let X be a smooth projective variety. If $\kappa(X) > 0$, then the Iitaka fibration of X is a morphism of projective varieties $f\colon X' \to Y$, with X' birational to X and Y of dimension $\kappa(X)$, such that the general fiber of f is smooth, irreducible, of zero Kodaira dimension. The Iitaka fibration is determined only up to birational equivalence. Since we are interested in questions of a birational nature, we usually assume that $X = X'$ and that Y is smooth.

X has *maximal Albanese dimension* if $\dim(\mathrm{alb}_X(X)) = \dim(X)$.

Proposition 2.1. *Let X be a smooth projective variety of maximal Albanese dimension, and let $f\colon X \to Y$ be the Iitaka fibration (assume Y smooth). Denote by $f_*\colon \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ the homomorphism induced by f and consider the commutative diagram:*

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{alb}_X} & \mathrm{Alb}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} \\ Y & \xrightarrow{\mathrm{alb}_Y} & \mathrm{Alb}(Y). \end{array}$$

Then:

a) *Y has maximal Albanese dimension;*
b) *f_* is surjective and $\ker f_*$ is connected of dimension $\dim(X) - \kappa(X)$;*
c) *there exists an abelian variety P isogenous to $\ker f_*$ such that the general fiber of f is birational to P.*

Proof. The dual of the differential of f_* at 0 is the pull–back map $H^0(Y, \Omega^1_Y) \to H^0(X, \Omega^1_X)$, which is clearly injective. It follows that f_* is surjective. Denote by K the connected component of $\ker f_*$ that contains 0 and set $A := \mathrm{Alb}(X)/K$. The abelian variety A is a finite étale cover of $\mathrm{Alb}(Y)$, and $f\colon X \to Y$ factors through the induced map $Y \times_{\mathrm{Alb}(Y)} A \to Y$, which is a connected étale cover of the same degree as $A \to \mathrm{Alb}(Y)$. Since the fibers of f are connected by assumption, it follows that $A \to \mathrm{Alb}(Y)$ is an isomorphism and the fibers of f_* are connected.

Let F be a general fiber of f. By Theorem 1 of [Ka], the Albanese map of F is surjective, and therefore the image of F via alb_X is a translate of an abelian subvariety of $\mathrm{Alb}(X)$ which we again denote by K. K is contained in $\ker f_*$ and is independent of F, since F moves in a continuous system. Denote by A the quotient abelian variety $\mathrm{Alb}(X)/K$. The induced map $X \to A$ is constant on the general fiber of f and thus induces a rational map $\phi\colon Y \to A$. Since A is an abelian

variety, ϕ is actually a morphism and $\phi(Y)$ generates A by construction. It follows that $\dim(A) \leq q(Y) = \dim \mathrm{Alb}(X) - \dim(\ker f_*)$, namely $\dim(K) \geq \dim(\ker f_*)$. It follows that K is equal to $\ker f_*$. By the theorem on the dimension of the fibers of a morphism, if X has maximal Albanese dimension, then Y also has maximal Albanese dimension, and $\ker f_*$ has dimension $\dim(X) - \dim(Y) = \dim(X) - \kappa(X)$. Thus $q(F) \geq \dim(F)$, and Corollary 2 of [Ka] implies that the Albanese map of F is a birational morphism. So the Albanese variety of F is isogenous to $\ker f_*$ and, in particular, it does not depend on F, since F moves in a continuous system. □

2.2. **Fourier Mukai Transforms.** Let A be an abelian variety, and denote the corresponding dual abelian variety by $\hat{A}$. Let $\mathcal{P}$ be the normalized Poincaré bundle on $A \times \hat{A}$ and let $\pi_{\hat{A}}\colon A \times \hat{A} \to \hat{A}$ be the projection. For any point $y \in \hat{A}$, let $\mathcal{P}_y$ denote the associated topologically trivial line bundle. Define the functor $\hat{\mathcal{S}}$ of $\mathcal{O}_A$-modules into the category of $\mathcal{O}_{\hat{A}}$-modules by

$$\hat{\mathcal{S}}(M) = \pi_{\hat{A},*}(\mathcal{P} \otimes \pi_A^* M).$$

The derived functor $R\hat{\mathcal{S}}$ of $\hat{\mathcal{S}}$ then induces an equivalence of categories between the two derived categories $D(A)$ and $D(\hat{A})$. In fact, if $R\mathcal{S}$ is the analogous functor on $D(\hat{A})$, one has (cf. [Mu]): *There are isomorphisms of functors:*

$$R\mathcal{S} \circ R\hat{\mathcal{S}} \cong (-1_A)^*[-g]$$

and

$$R\hat{\mathcal{S}} \circ R\mathcal{S} \cong (-1_{\hat{A}})^*[-g],$$

where $[-g]$ *denotes "shift the complex* g *places to the right".*

The *index theorem* (I.T.) is said to hold for a coherent sheaf $\mathcal{F}$ on A if there exists an integer $i(\mathcal{F})$ such that for all $j \neq i(\mathcal{F})$, $H^j(A, \mathcal{F} \otimes P) = 0$ for all $P \in \mathrm{Pic}^0(A)$. The *weak index theorem* (W.I.T.) holds for a coherent sheaf $\mathcal{F}$ if there exists an integer, which we again denote by $i(\mathcal{F})$, such that for all $j \neq i(\mathcal{F})$, $R^j\hat{\mathcal{S}}(\mathcal{F}) = 0$. It is easily seen that the I.T. implies the W.I.T. If $\mathcal{F}$ satisfies the W.I.T., we denote the coherent sheaf $R^{i(\mathcal{F})}\hat{\mathcal{S}}(\mathcal{F})$ on $\hat{A}$ by $\hat{\mathcal{F}}$. By Corollary 2.4 of [Mu], $\hat{\mathcal{F}}$ also satisfies W.I.T. and has index $\dim(A) - i(\mathcal{F})$.

Here are some consequences of the theory of [Mu]:

Lemma 2.2. *(Non-vanishing) If* $\mathcal{F}$ *is a coherent sheaf on* A *such that* $h^i(\mathcal{F} \otimes P) = 0$ *for all* $i \geq 0$ *and all* $P \in \mathrm{Pic}^0(A)$*, then* $\mathcal{F} = 0$*.*

Proof. Follows immediately from [Mu, Theorem 2.2]. □

Proposition 2.3. *Let* $\psi\colon \mathcal{F} \hookrightarrow \mathcal{G}$ *be an inclusion of coherent sheaves on* A *inducing isomorphisms* $H^i(A, \mathcal{F} \otimes P) \to H^i(A, \mathcal{G} \otimes P)$ *for all* $i \geq 0$ *and all* $P \in \mathrm{Pic}^0(A)$*. Then* ψ *is an isomorphism of sheaves.*

Proof. Let $\mathcal{K}$ be the cokernel of $\psi\colon \mathcal{F} \hookrightarrow \mathcal{G}$. Clearly $h^i(A, \mathcal{K}\otimes P) = 0$ for all $i \geq 0$ and all $P \in \mathrm{Pic}^0(A)$. Therefore, $\mathcal{K} = 0$ by Lemma 2.2, and ψ is an isomorphism. □

Proposition 2.4. *Let $\mathcal{F}$ be a coherent sheaf on A such that for all $P \in \mathrm{Pic}^0(A)$ we have $h^0(\mathcal{F}\otimes P) = 1$ and $h^i(\mathcal{F}\otimes P) = 0$ for all $i > 0$. Then $\mathcal{F}$ is supported on an abelian subvariety of A.*

Proof. $\mathcal{F}$ satisfies the I.T., $M := \hat{\mathcal{F}}$ is a line bundle on $\mathrm{Pic}^0(A) = \hat{A}$ that satisfies the W.I.T. with index $i(M) = \dim(A)$, and $\hat{M} = (-1_A)^*\mathcal{F}$. Any line bundle with $i(M) = \dim(A)$ is negative semidefinite. It is well known that there exists a morphism of abelian varieties $b\colon \hat{A} \to A'$ such that $M = b^*M'$ for some negative definite line bundle M' on A'. It follows that $\hat{M}$ and hence $\mathcal{F}$ are supported on the image of $b^*\colon \hat{A}' \to A$. □

We also recall the following result from [Hac]:

Theorem 2.5. *Let X be a smooth complex projective variety of dimension n, let A be an abelian variety of dimension $n+1$, and let $f\colon X \to A$ be a morphism generically finite onto its image. Assume that:*

a) $h^0(X, \Omega^i_X) = \binom{n+1}{i}$ *for* $0 \leq i \leq n$;
b) $h^i(X, \omega_X\otimes(f^*P)) = 0$ *for all* $\mathcal{O}_A \neq P \in \mathrm{Pic}^0(A)$ *and all* $i > 0$.

Then A is principally polarized and $f(X)$ is a theta divisor.

Proof. Cf. [Hac], Corollary 3.4. □

2.3. **Direct images of dualizing sheaves.** The following Theorem summarizes the most important facts on direct images of adjoint bundles. We refer to [Ko3], (10.1.5) for the definition of "klt" (Kawamata log terminal).

Theorem 2.6. *Let $f\colon X \to Y$ be a surjective map of projective varieties, X smooth and Y normal. Let M be a line bundle on X such that $M \equiv f^*L + \Delta$, where L is a $\mathbb{Q}-$divisor on Y and (X, Δ) is klt. Then*

a) $R^jf_*(\omega_X\otimes M)$ *is torsion free for* $j \geq 0$;
b) *If L is nef and big, then* $H^i(Y, R^jf_*(\omega_X\otimes M)) = 0$ *for* $i > 0$, $j \geq 0$;
c) $R^if_*\omega_X$ *is zero for* $i > \dim(X) - \dim(Y)$.

Proof. Statements a) and b) correspond to Corollary 10.15 of [Ko3]. Statement c) follows from Theorem 2.1 of [Ko1]. □

The following observation is an easy consequence of the previous results.

Lemma 2.7. *Let $f\colon X \to Y$ be a surjective map of projective varieties, with X smooth and Y normal, and let M be a line bundle on X such*

$M \equiv f^*L + \Delta$, where L is a nef e big $\mathbb{Q}-$divisor on Y and (X, Δ) is klt. If $g\colon Y \to Z$ is any morphism with Z projective, then

$$H^i(Y, R^j f_*(K_X + M)) = H^i(Z, g_*(R^j f_*(K_X + M))) \quad i, j \geq 0.$$

Proof. Clearly we may replace Z by the image of g and assume that g is onto. In addition, may assume that Z is normal, since g factors through the normalization $\nu\colon Z' \to Z$ and $R^i\nu_*\mathcal{F} = 0$ for any coherent sheaf $\mathcal{F}$ on Z' and $i > 0$. By the Leray spectral sequence, to prove the claim it is enough to show that the sheaves $R^i g_*(R^j f_*(\omega_X \otimes M))$ are zero for $j \geq 0$ and $i > 0$. Fix an ample divisor H on Z such that:

a) $R^i g_*(R^j f_*(\omega_X \otimes M)) \otimes H$ is generated by global sections;

b) $H^k(Z, R^i g_*(R^j f_*(\omega_X \otimes M)) \otimes H) = 0$ for $k > 0$.

The Leray spectral sequence that computes $H^i(Y, R^j f_*(\omega_X \otimes M) \otimes g^*H)$ degenerates at E_2 by condition b), and we have isomorphisms between $H^i(Y, R^j f_*(\omega_X \otimes M) \otimes g^*H)$ and $H^0(Z, R^i g_*(R^j f_*(\omega_X \otimes M)) \otimes H)$. For $i > 0$, the former group vanishes by Theorem 2.6, b), since $L \otimes g^*H$ is again nef and big. Now for $i > 0$ the sheaf $R^i g_*(R^j f_*(\omega_X \otimes M)) \otimes H$ is zero by condition a), and thus $R^i g_*(R^j f_*(\omega_X \otimes M))$ is also zero. $\square$

We use the techniques of 2.2 to reprove a particular case of a result of Kollàr, ([Ko3], Theorem 14.7). We remark that in [Ko3] the hypothesis that Y be of maximal Albanese dimension is replaced by the less restrictive hypothesis that Y has generically large fundamental group. However the proof is more technical.

Theorem 2.8. (Kollàr) *Let $f\colon X \to Y$ be a surjective map of projective varieties, X smooth and Y normal. Let M be a line bundle on X such that $M \equiv f^*L + \Delta$, where L is a nef and big $\mathbb{Q}$-divisor on Y and (X, Δ) is klt. If Y is of maximal Albanese dimension, then for every $j \geq 0$:*

a) *$h^0(Y, R^j f_*(\omega_X \otimes M) \otimes P)$ is independent of $P \in \mathrm{Pic}^0(Y)$;*

b) *$H^0(Y, R^j f_*(\omega_X \otimes M) \otimes P) = 0$ iff $R^j f_*(\omega_X \otimes M) = 0$.*

Proof. If $R^j f_*(\omega_X \otimes M) = 0$, then clearly $H^0(Y, R^j f_*(\omega_X \otimes M) \otimes P) = 0$ for all $P \in \mathrm{Pic}^0(Y)$.

Suppose now that $R^j f_*(\omega_X \otimes M) \neq 0$. The sheaf $R^j f_*(\omega_X \otimes M)$ is torsion free by Theorem 2.6, a). Let $a\colon Y \to A := \mathrm{Alb}(Y)$ be the Albanese map of Y, which is generically finite by assumption. The sheaf $a_*(R^j f_*(\omega_X \otimes M))$ is non-zero, supported on $a(Y)$. By Lemma 2.7, for every $P \in \mathrm{Pic}^0(Y)$ and for every $i \geq 0$ we have isomorphisms $H^i(A, a_*(R^j f_* \omega_X \otimes M) \otimes P) \cong H^i(Y, R^j f_*(\omega_X \otimes M) \otimes P)$. If $i > 0$, then the latter group vanishes by Theorem 2.6, b), and hence

$$h^0(Y, R^j f_*(\omega_X \otimes M) \otimes P) = \chi(R^j f_*(\omega_X \otimes M) \otimes P) = \chi(R^j f_*(\omega_X \otimes M))$$

is independent of P. In addition, by Lemma 2.2 there is $P \in \mathrm{Pic}^0(Y)$ such that $h^0(A, a_*(R^j f_*(\omega_X \otimes M)) \otimes P) > 0$. So $h^0(Y, R^j f_*(\omega_X \otimes M) \otimes P)$ $= h^0(A, a_*(R^j f_*(\omega_X \otimes M) \otimes P)) > 0$ for every $P \in \mathrm{Pic}^0(Y)$. $\square$

We also need to understand the behavior of direct images of dualizing sheaves with respect to the Stein factorization.

Let Y be a smooth variety. For a coherent sheaf $\mathcal{F}$ on Y, we denote by $\mathcal{F}^{\vee}$ the *dual sheaf* $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y)$. A sheaf $\mathcal{F}$ is said to be *reflexive* if the natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism. Following [OSS], we say that $\mathcal{F}$ is *normal* iff for every open set $U \subset Y$ and every closed subset $C \subset Y$ of codimension > 1 the restriction map $\mathcal{F}(U) \to \mathcal{F}(U \setminus C)$ is an isomorphism.

Proposition 2.9. *Let X, Y be smooth projective varieties, $f\colon X \to Y$ a generically finite morphism, and let $X \xrightarrow{h} Z \xrightarrow{g} Y$ be the Stein factorization of f. If $f_*\omega_X$ is locally free on Y, then g is a flat morphism and $g_*\mathcal{O}_Z = (f_*\omega_X)^{\vee} \otimes \omega_Y$.*

Proof. By the definition of Stein factorization, h is birational, g is finite and $g_*\mathcal{O}_Z = f_*\mathcal{O}_X$. The variety Z is normal, since X is normal, hence $g_*\mathcal{O}_Z$ is a normal sheaf, as defined above. Denote by ω_Z the dualizing sheaf of Z. By [Re], Corollary (8) p. 283, if $j\colon Z_0 \to Z$ is the inclusion of the smooth locus Z_0 of Z, one has $\omega_Z = j_*\omega_{Z_0}$. If we write $X_0 = h^{-1}(Z_0)$ and denote by $h_0\colon X_0 \to Z_0$ the restriction of h to Z_0, then we have $(h_0)_*\omega_{X_0} = \omega_{Z_0}$ since h_0 is a birational morphism of smooth varieties. This identification induces a sheaf map $\psi\colon f_*\omega_X \to g_*\omega_Z$ that is an isomorphism outside the image Σ of the singular locus of Z, which is a closed subset of Y of codimension > 1. The sheaf $f_*\omega_X$ is normal, since it is locally free and Y is smooth, while $g_*\omega_Z$ is normal by the definition of ω_Z and by the fact that g is finite. It follows that ψ is an isomorphism.

By [Hart], Chapter III, ex. 6.10b, there is an isomorphism $g_*(g^!\omega_Y) \cong (g_*\mathcal{O}_Z)^{\vee} \otimes \omega_Y$, where, for a coherent sheaf $\mathcal{F}$ of Y and a finite morphism $g\colon Z \to Y$, one defines $g^!\mathcal{F}$ as the coherent sheaf on Z corresponding to the $g_*\mathcal{O}_Z-$module $\mathcal{H}om_{\mathcal{O}_Y}(g_*\mathcal{O}_Z, \mathcal{F})$. Now $g^!\omega_Y = \omega_Z$, by [Hart], Chapter III, ex. 7.2, hence $g_*\omega_Z = (g_*\mathcal{O}_Z)^{\vee} \otimes \omega_Y$. Taking duals, one has $(g_*\mathcal{O}_Z)^{\vee\vee} \cong (g_*\omega_Z)^{\vee} \otimes \omega_Y \cong (f_*\omega_X)^{\vee} \otimes \omega_Y$. The sheaf $g_*\mathcal{O}_Z$ is normal and torsion free, since Z is normal, and thus it is reflexive by Lemma 1.1.12 of [OSS]. Thus we have $g_*\mathcal{O}_Z \cong (f_*\omega_X)^{\vee} \otimes \omega_Y$. In particular, $g_*\mathcal{O}_Z$ is locally free, namely g is flat. ☐

2.4. **Cohomological Support Loci.** Let $\pi\colon X \to A$ be a morphism from a smooth projective variety X to an abelian variety A. If $\mathcal{F}$ is a coherent sheaf on X, then one can define the *cohomological support loci* by

$$V^i(X, A, \mathcal{F}) := \{P \in \mathrm{Pic}^0(A) | h^i(X, \mathcal{F} \otimes \pi^*P) > 0\}.$$

In particular, if $\pi = \mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$, then we simply write

$$V^i(X, \mathcal{F}) := \{P \in \mathrm{Pic}^0(X) | h^i(X, \mathcal{F} \otimes P) > 0\}.$$

We sometimes write $V^i(\mathcal{F})$ instead of $V^i(X,\mathcal{F})$ if no confusion is likely to arise. We say that P is a *general point* of $V^i(X,A,\mathcal{F})$ if $V^i(X,A,\mathcal{F})$ is smooth at P and $h^i(\mathcal{F}\otimes P)$ is equal to the minimum of the function $h^i(\mathcal{F}\otimes -)$ on the component of $V^i(X,A,\mathcal{F})$ that contains P.

In [EL1], Ein and Lazarsfeld illustrate various examples in which the geometry of X can be recovered from information on the loci $V^i(\omega_X)$. The geometry of $V^i(\omega_X)$ is governed by the following:

Theorem 2.10. *(Generic Vanishing Theorem)*
Let X be a smooth projective variety. Then:

a) $V^i(\omega_X)$ *has codimension* $\geq i-(\dim(X)-\dim(\mathrm{Alb}(X))$*;*
b) *every irreducible component of* $V^i(X,\omega_X)$ *is a translate of a subtorus of* $\mathrm{Pic}^0(X)$ *by a torsion point;*
c) *let T be an irreducible component of $V^i(\omega_X)$, let $P\in T$ be a point such that $V^i(\omega_X)$ is smooth at P, and let $v\in H^1(X,\mathcal{O}_X)\cong T_P\mathrm{Pic}^0(X)$. If v is not tangent to T, then the sequence*
$$H^{i-1}(X,\omega_X\otimes P)\xrightarrow{\cup v}H^i(X,\omega_X\otimes P)\xrightarrow{\cup v}H^{i+1}(X,\omega_X\otimes P)$$
is exact. Moreover, if P is a general point of T and v is tangent to T then both maps vanish;
d) *if X has maximal Albanese dimension, then there are inclusions:*
$$V^0(\omega_X)\supseteq V^1(\omega_X)\supseteq\cdots\supseteq V^n(\omega_X)=\{\mathcal{O}_X\}.$$

Proof. Statement a) is Theorem 1 of [GL1]. For statement b), the fact that the components of $V^i(\omega_X)$ are translates of abelian subvarieties follows from Theorem 0.1 of [GL2] and the fact that the translation is by a torsion point follows from Theorem 4.2 of [Si]. Statement c) follows from Theorem 1.2 of [EL1]. Statement d) is Lemma 1.8 of [EL1]. $\square$

Remark 2.11. *If $\pi\colon X\to A$ is a morphism to an abelian variety, then the loci $V^i(X,A,\omega_X)$ satisfy properties analogous to a) … d) of Theorem 2.10 (cf.* [EL1], *Remark 1.6).*

The results that follow are refinements of Theorem 2.10 for the case of a variety of maximal Albanese dimension with $\chi(\omega_X)=0$ or 1.

Proposition 2.12. *Let X be a smooth projective variety of maximal Albanese dimension with $\chi(\omega_X)=1$ and $q(X)\geq\dim(X)+1$. If $P\in\mathrm{Pic}^0(X)$ is an isolated point of $V^1(\omega_X)$, then $P=\mathcal{O}_X$.*

Proof. We write $n:=\dim(X)$ and $q:=q(X)\geq n+1$. For $n=1$ the claim follows by Serre duality, hence we may assume $2\leq n$. We will assume that $P\neq\mathcal{O}_X$ and show that $h^{n-1}(P^{-1})=h^1(\omega_X\otimes P)=0$. We remark that since $P\neq\mathcal{O}_X$, one has $h^0(P^{-1})=0$. Let k be the greatest integer such that $h^k(\omega_X\otimes P)>0$. Then by Theorem 2.10, d), P is an

isolated point of $V^j(\omega_X)$ for $1 \leq j \leq k$. By Theorem 2.10, c) and Serre duality, for every $0 \neq v \in H^1(\mathcal{O}_X)$ the complex:

$$0 \to H^0(P^{-1}) \stackrel{\cup v}{\to} H^1(P^{-1}) \stackrel{\cup v}{\to} \dots \stackrel{\cup v}{\to} H^n(P^{-1})$$

is exact except at the last term. As v varies, the above complexes fit together to give a complex of vector bundles on $\mathbb{P} := \mathbb{P}(H^1(\mathcal{O}_X)^\vee))$ (see [EL1], proof of Thm. 3, for a similar argument):

$$0 \to H^0(P^{-1}) \otimes \mathcal{O}_{\mathbb{P}}(-n) \to H^1(P^{-1}) \otimes \mathcal{O}_{\mathbb{P}}(-n+1) \to \dots$$
$$\dots \to H^n(P^{-1}) \otimes \mathcal{O}_{\mathbb{P}}$$

which is again exact except at the last term, since exactness can be checked fibrewise. For $\mathcal{O}_X \neq Q$ in an appropriate neighborhood of $\mathcal{O}_X$, one has that $h^i(P^{-1} \otimes Q) = 0$ for $i < n$, hence $h^n(P^{-1} \otimes Q) = \chi(\omega_X \otimes P \otimes Q^{-1}) = \chi(\omega_X) = 1$. It follows by [GL2] Corollary 3.3, that for every $0 \neq v \in H^1(X, \mathcal{O}_X)$, the cokernel of the map $H^{n-1}(P^{-1}) \stackrel{\cup v}{\to} H^n(P^{-1})$ is $1-$dimensional. Therefore the cokernel of the map of vector bundles

$$H^{n-1}(P^{-1}) \otimes \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow H^n(P^{-1}) \otimes \mathcal{O}_{\mathbb{P}}$$

is a line bundle $\mathcal{O}_{\mathbb{P}}(d)$ with $d \geq 0$. Let $\mathcal{K}^\bullet$ be the complex of vector bundles given by the exact sequence

$$0 \to H^0(P^{-1}) \otimes \mathcal{O}_{\mathbb{P}}(-n) \to H^1(P^{-1}) \otimes \mathcal{O}_{\mathbb{P}}(-n+1) \to \dots$$
$$\dots \to H^n(P^{-1}) \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(d) \to 0.$$

For any i, one can consider the two spectral sequences associated with the hypercohomology of the complex $\mathcal{K}^\bullet \otimes \mathcal{O}_{\mathbb{P}}(i)$. Since the complex is exact, we have $'E_1^{s,t} = 0$ for all s, t, and thus $'E_\infty^{s,t} = 0$. On the other hand, for $i = -1$, one has that $''E_1^{s,t} = 0$ except for $s = 0$ and $t = n+1$, in which case $''E_1^{0,n+1} = H^0(\mathcal{O}_{\mathbb{P}}(d-1))$ (recall that $h^0(P^{-1}) = 0$ by assumption). Comparing the two spectral sequences, one sees that $H^0(\mathcal{O}_{\mathbb{P}}(d-1)) = 0$ and hence $d = 0$. For $i = 0$ the only non zero terms are $''E_1^{0,n} = H^n(P^{-1}) \otimes H^0(\mathcal{O}_{\mathbb{P}})$ and $''E_1^{0,n+1} = H^0(\mathcal{O}_{\mathbb{P}})$, therefore $h^n(P^{-1}) = 1$. In particular, the map $H^n(P^{-1}) \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}$ is an isomorphism. For $i = 1$ the only non zero terms are $''E_1^{0,n-1} = H^{n-1}(P^{-1}) \otimes H^0(\mathcal{O}_{\mathbb{P}})$, $''E_1^{0,n} = H^n(P^{-1}) \otimes H^0(\mathcal{O}_{\mathbb{P}}(1))$ and $''E_1^{0,n+1} = H^0(\mathcal{O}_{\mathbb{P}}(1))$. The differential $d_1 \colon ''E_1^{0,n} \to ''E_1^{0,n+1}$ is an isomorphism and hence $''E_1^{0,n-1} = 0$ i.e. $H^{n-1}(P^{-1}) = 0$.

$\square$

Proposition 2.13. *Let X be a smooth projective variety of maximal Albanese dimension such that $\chi(\omega_X) = 0$. Let T be a component of $V^0(\omega_X)$ and let $W \subset H^1(\mathcal{O}_X)$ be a linear subspace complementary to the tangent space to T (recall that T is a translate of an abelian subvariety of* $\mathrm{Pic}^0(X)$*). Let $P \in T$ be a point such that $h^i(\omega_X \otimes P) = 0$*

for $i > \dim(W)$ and such that P is a smooth point of $V^0(\omega_X)$. Then the natural map induced by cup product:

$$H^0(\omega_X \otimes P) \otimes \wedge^j W \to H^j(\omega_X \otimes P)$$

is an isomorphism for every $j \geq 0$.

Before proving the Proposition, we wish to point out that the assumptions on P are satisfied if $P \in T$ is general (cf. [EL1], proof of Theorem 3).

Proof. Notice that by Theorem 2.10, a), the assumptions on X imply that $V^0(\omega_X)$ is a proper subset of $\mathrm{Pic}^0(X)$. Write $n := \dim(X)$ and $w := \dim(W)$. As explained in the proof of Theorem 3 of [EL1], T is a component of $V^w(\omega_X)$ and $T \not\subset V^i(\omega_X)$ for $i > w$. By Theorem 2.10, c), the assumptions on P imply that the complex $D(v)$:

$$\dots H^{j-1}(\omega_X \otimes P) \xrightarrow{\cup v} H^j(\omega_X \otimes P) \xrightarrow{\cup v} H^{j+1}(\omega_X \otimes P) \dots$$

is exact for all $v \in W \subset H^1(X, \mathcal{O}_X)$. As we have already seen in the proof of Proposition 2.12, the complexes $D(v)$ fit together to give an exact complex $\mathcal{K}^\bullet$ of vector bundles on $\mathbb{P} := \mathbb{P}(W)$:

$$0 \to H^0(\omega_X \otimes P) \otimes \mathcal{O}_{\mathbb{P}}(-w) \to H^1(\omega_X \otimes P) \otimes \mathcal{O}_{\mathbb{P}}(-w+1) \to \dots$$
$$\dots \to H^n(\omega_X \otimes P) \otimes \mathcal{O}_{\mathbb{P}}(-w+n) \to 0.$$

Similarly, there is an exact sequence $\mathcal{K}_0^\bullet$ of vector bundles on $\mathbb{P}$:

$$\dots \to H^0(\omega_X \otimes P) \otimes \wedge^{j-1} W \otimes \mathcal{O}_{\mathbb{P}}(-w+j-1) \to H^0(\omega_X \otimes P) \otimes \wedge^j W \otimes \mathcal{O}_{\mathbb{P}}(-w+j) \to$$
$$\to H^0(\omega_X \otimes P) \otimes \wedge^{j+1} W \otimes \mathcal{O}_{\mathbb{P}}(-w+j+1) \to \dots$$

There is a map of complexes $\mathcal{K}_0^\bullet \to \mathcal{K}^\bullet$ induced by

$$H^0(\omega_X \otimes P) \otimes \wedge^j W \otimes \mathcal{O}_{\mathbb{P}}(-w+j) \hookrightarrow H^0(\omega_X \otimes P) \otimes \wedge^j H^1(X, \mathcal{O}_X) \otimes \mathcal{O}_{\mathbb{P}}(-w+j)$$
$$\to H^j(\omega_X \otimes P) \otimes \mathcal{O}_{\mathbb{P}}(-w+j)$$

Clearly for $j = 0$ there is an isomorphism

$$H^0(\omega_X \otimes P) \cong H^0(\omega_X \otimes P) \otimes \wedge^0 W.$$

Proceeding by induction, assume $1 \leq j \leq \dim(X)$ and

$$H^l(\omega_X \otimes P) \cong H^0(\omega_X \otimes P) \otimes \wedge^l W$$

for all $l < j$. Tensoring by $\mathcal{O}_{\mathbb{P}}(-j)$ and taking cohomology, one gets:

$$\begin{array}{ccc}
\dots & & \dots \\
\downarrow & & \downarrow \\
H^0(\omega_X \otimes P) \otimes \wedge^{j-1} W \otimes H^{w-1}(\mathcal{O}_{\mathbb{P}}(-w-1)) & \xrightarrow{\sim} & H^{j-1}(\omega_X \otimes P) \otimes H^{w-1}(\mathcal{O}_{\mathbb{P}}(-w-1)) \\
\downarrow & & \downarrow \\
H^0(\omega_X \otimes P) \otimes \wedge^j W \otimes H^{w-1}(\mathcal{O}_{\mathbb{P}}(-w)) & \longrightarrow & H^j(\omega_X \otimes P) \otimes H^{w-1}(\mathcal{O}_{\mathbb{P}}(-w)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

If the vertical rows are exact, then the required isomorphism follows from the five lemma . Consider the spectral sequence associated to the complex $\mathcal{K}^\bullet \otimes \mathcal{O}_{\mathbb{P}}(-j)$. We have $'E_1^{s,t} = 0$ for any s, t, thus $'E_\infty^{s,t} = 0$. On the other hand for $1 \leq j \leq \dim(X)$ one has $''E_1^{s,t} = 0$ if $s \neq w-1$ and $s \neq 0$. If $s = 0$ then $''E_1^{0,t} = H^t(\omega_X \otimes P) \otimes H^0(\mathcal{O}_{\mathbb{P}}(-w+t-j))$. For $0 \leq t \leq w$ this group is 0 as $-w+t-j < 0$, while for $t > w$ it is 0 since $h^t(\omega_X \otimes P) = 0$ by assumption. Comparing spectral sequences one sees that the sequence

$$\ldots \longrightarrow H^{j-1}(\omega_X \otimes P) \otimes H^{w-1}(\mathcal{O}_{\mathbb{P}}(-w-1)) \longrightarrow H^j(\omega_X \otimes P) \otimes H^{w-1}(\mathcal{O}_{\mathbb{P}}(-w)) \longrightarrow \ldots$$

is exact. The exactness of the first vertical row can be shown in the same way. $\square$

The following Corollary is due to Ein–Lazarsfeld (cf. [EL2]).

Corollary 2.14. *Let X be a smooth projective variety of maximal Albanese dimension such that $\chi(\omega_X) = 0$. If $P \in \mathrm{Pic}^0(X)$ is an isolated point of $V^0(\omega_X)$ then $P = \mathcal{O}_X$.*

Proof. Write as usual $n := \dim X$, $q := q(X)$. By Proposition 2.13 we have $H^q(\omega_X \otimes P) \cong \wedge^q H^1(\mathcal{O}_X) \otimes H^0(\omega_X \otimes P) \neq 0$. This implies $q \leq n$, hence $q = n$, since X has maximal Albanese dimension. In particular we have $0 \neq h^n(\omega_X \otimes P) = h^0(P^{-1})$, implying $P = \mathcal{O}_X$. $\square$

In some cases it is possible to obtain more precise information on the linear systems $|mK_X + P|$ for $m > 1$ and $P \in V^0(\omega_X^m)$.

Proposition 2.15. *Let X be a smooth projective variety and let $f\colon X \to Y$ be the Iitaka fibration (assume Y smooth). Fix $m \geq 2$ and $Q \in \mathrm{Pic}^\tau(X)$. Then:*

$$h^0(mK_X + Q + P) = h^0(mK_X + Q) \text{ for all } P \in \mathrm{Pic}^0(Y).$$

Proof. The statement is trivial if $\kappa(X) = \dim(Y) = 0$, hence we may assume $\kappa(X) > 0$.

Step 1. *For all $P \in \mathrm{Pic}^0(Y)$, $\kappa(K_X + P) \geq \kappa(X)$.*

Arguing as in the proof of [CH1, Lemma 2.1], one can show that if $m \geq 2$ then $h^0(mK_X) = h^0(mK_X + P)$ for all $P \in \mathrm{Pic}^\tau(Y)$. Since $\mathrm{Pic}^\tau(Y)$ is dense in $\mathrm{Pic}^0(Y)$, by semicontinuity one has that $h^0(m(K_X + P)) = h^0(mK_X + mP) \geq h^0(mK_X)$ for every $P \in \mathrm{Pic}^0(Y)$.

Step 2. *Let $P \in \mathrm{Pic}^0(Y)$. We have $h^0(mK_X + P + Q) = h^0(mK_X + P + R + Q)$ for all $R \in \mathrm{Pic}^\tau(Y)$*

This follows by a procedure analogous to [CH1, Lemma 2.1]. We illustrate this for $m = 2$. Write $S := \mathrm{Alb}(Y)$ and denote by $\pi\colon X \to S$ the composition of f and of the Albanese map of Y. Fix H an ample line bundle on S and $R \in \mathrm{Pic}^\tau(Y)$. By Step 1, for r sufficiently big and divisible we may pick a divisor:

$$B \in |r(K_X+P) - \pi^*H| = |r(K_X+P+Q+R) - \pi^*H| = |r(K_X+P+Q) - \pi^*H|.$$

Possibly replacing X by an appropriate birational model, we may assume that B has normal crossings support and that (as in [CH1, Lemma 2.1])

$$\lfloor \frac{B}{r} \rfloor \subset Bs|2K_X + P + Q| \cap Bs|2K_X + P + Q + R|.$$

Let

$$L := K_X - \lfloor \frac{B}{r} \rfloor \equiv \frac{\pi^* H}{r} + \{\frac{B}{r}\}$$

i.e. L is numerically equivalent to the pull back of a nef and big $\mathbb{Q}-$divisor plus a klt divisor. Then comparing base loci as in [CH1, Lemma 2.1]

$$h^0(2K_X + P + Q) = h^0(K_X + L + P + Q) =$$
$$= h^0(\pi_*(\omega_X \otimes L \otimes P \otimes Q)) = \chi(\pi_*(\omega_X \otimes L \otimes P \otimes Q)) =$$
$$= \chi(\pi_*(\omega_X \otimes L \otimes P \otimes Q \otimes R)) = h^0(\pi_*(\omega_X \otimes L \otimes P \otimes Q \otimes R)) =$$
$$= h^0(K_X + L + P + Q + R) = h^0(2K_X + P + Q + R),$$

where the third and the fifth equality follow from Theorem 2.6, b).

Step 3. *For all $P \in \mathrm{Pic}^0(Y)$, $h^0(mK_X + P + Q) = h^0(mK_X + Q)$.*

Let M be the maximum of the function $h^0(mK_X + Q + P)$ for $P \in \mathrm{Pic}^0(Y)$ and let $P_0 \in \mathrm{Pic}^0(Y)$ be such that $h^0(mK_X + Q + P_0) = M$. By Step 2, $h^0(mK_X + Q + P_0 + R) = M$ for all $R \in \mathrm{Pic}^\tau(Y)$. Since $\mathrm{Pic}^\tau(Y)$ is dense in $\mathrm{Pic}^0(Y)$, by semicontinuity we have $h^0(mK_X + Q + P) \geq M$ for all $P \in \mathrm{Pic}^0(Y)$, hence $h^0(mK_X + Q + P) = M$ for all $P \in \mathrm{Pic}^0(Y)$. $\square$

In §6 we will need following result, which is due to Ein and Lazarsfeld.

Lemma 2.16. *Let X be a variety such that $\chi(\omega_X) = 0$ and such that $\mathrm{alb}_X \colon X \to \mathrm{Alb}(X)$ is surjective and generically finite. Let T be an irreducible component of $V^0(\omega_X)$, and let $\pi_E \colon X \to E := \mathrm{Pic}^0(T)$ be the morphism induced by the map $\mathrm{Alb}(X) = \mathrm{Pic}^0(\mathrm{Pic}^0(X)) \to E$ corresponding to the inclusion $T \hookrightarrow \mathrm{Pic}^0(X)$.*

Then there exists a divisor $D \prec R := Ram(\mathrm{alb}_X) = K_X$, vertical with respect to $g := \pi_E \circ \mathrm{alb}_X$, such that for general $P \in T$, $F := R - D$ is a fixed divisor of each of the linear series $|K_X + P|$.

Proof. This is [EL2], Lemma D. We thank Ein and Lazarsfeld for allowing us to reproduce their proof.

Let $n = \dim(X) = q(X)$, and let k be the codimension of T in $\mathrm{Pic}^0(X)$. Choose a basis

$$\omega_1, ..., \omega_n \in H^0(X, \Omega^1_X)$$

such that $\omega_{k+1}, ..., \omega_n$ are pull backs under g of a basis for $H^0(E, \Omega^1_E)$. In particular we may assume that $\omega_{k+1}, ..., \omega_n$ are conjugate to a basis of the tangent space to T. There is a homomorphism:

$$\alpha : \mathcal{O}_X = g^* \Omega^{n-k}_E \longrightarrow \Omega^{n-k}_X$$

defined by the section $\omega_{k+1} \wedge ... \wedge \omega_n \in H^0(X, \Omega_X^{n-k})$. α vanishes at a point $x \in X$ if and only if g is not smooth at x. Let $D \subset X$ be the divisor along which α vanishes. There is a diagram:

$$\begin{array}{ccccccccc} & & \mathcal{O}_X & & & & & & \\ & & \Big\downarrow{\scriptstyle D} & & & & & & \\ 0 & \longrightarrow & \mathcal{O}_X(D) & \xrightarrow{\bar{\alpha}} & \Omega_X^{n-k} & \longrightarrow & coker(\bar{\alpha}) & \longrightarrow & 0. \end{array}$$

An easy local computation shows that $coker(\bar{\alpha})$ is torsion free. Fix a general point $P \in T$. By the proof of Theorem 3 of [EL1] (see also Proposition 2.13) one sees that $H^0(X, \Omega_X^{n-k} \otimes P) \neq 0$. We claim that $\bar{\alpha}$ induces an isomorphism:

$$H^0(X, \mathcal{O}_X(D) \otimes P) = H^0(X, \Omega_X^{n-k} \otimes P).$$

Fix a non-zero section $\eta \in H^0(X, \Omega_X^{n-k} \otimes P)$. Since $coker(\bar{\alpha})$ is a torsion free sheaf, it suffices to show that at a general poit $x \in X$, $\eta(x)$ lies in the subspace of $\Omega_X^{n-k} \otimes P(x) \cong \Omega_X^{n-k}(x)$ spanned by $\omega_{k+1}(x) \wedge ... \wedge \omega_n(x)$. By Theorem 2.10 for all $k+1 \leq i \leq n$, one has $\eta \wedge \omega_i = 0 \in H^0(X, \Omega_X^n \otimes P)$. Since $\omega_1(x), ..., \omega_n(x)$ span $T_x^*(X)$ at a general point, $\eta(x)$ must be a multiple of $\omega_{k+1} \wedge ... \wedge \omega_n$.

Consider finally the commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{R} & \Omega_X^n \otimes P \\ {\scriptstyle D}\Big\downarrow & & \Big\uparrow{\scriptstyle \wedge\omega_1 \wedge ... \wedge \omega_k} \\ \mathcal{O}_X(D) \otimes P & \xrightarrow{\bar{\alpha}} & \Omega_X^{n-k} \otimes P. \end{array}$$

The top row is multiplication by the ramification divisor $R = \{\omega_1 \wedge ... \wedge \omega_n = 0\}$, therefore $D \prec R$. As we have seen above, $\bar{\alpha}$ gives an isomorphism on global sections. By the proof of Theorem 3 of [EL1] (cf. Proposition 2.13) one sees that the right hand side vertical homomorphism also induces an isomorphism on global sections. Therefore

$$H^0(\mathcal{O}_X(D) \otimes P) = H^0(K_X \otimes P).$$

In other words $F = R - D$ is a fixed divisor of the linear series $|K_X + P|$. □

Finally, we often use the following observation to give lower bounds on the dimension of the series $|mK_X + P|$.

Lemma 2.17. *Let X be a smooth projective variety, let L and M be line bundles on X, and let $T \subset \mathrm{Pic}^0(X)$ be an irreducible subvariety of dimension t. If for all $P \in T$, $\dim |L + P| \geq a$ and $\dim |M - P| \geq b$, then $\dim |L + M| \geq a + b + t$*

Proof. Denote by $\mathcal{P}$ the restriction to $X \times T$ of the Poincaré line bundle on $X \times \mathrm{Pic}^0(X)$ and by p_i, $i = 1, 2$, the projections of $X \times T$ on the $i-$th factor. Set $\mathcal{V}_1 := p_{2*}(\mathcal{P} \otimes p_1^* M)$ and $\mathcal{V}_2 := p_{2*}(\mathcal{P}^{-1} \otimes L)$ and denote by ρ_i

the generic rank of $\mathcal{V}_i$, $i=1,2$. Denote by $\psi\colon \mathcal{V}_1\otimes\mathcal{V}_2 \to p_{2*}(p_1^*(L\otimes M))$ the sheaf map induced by the multiplication map

$$(p_1^*L\otimes\mathcal{P})\otimes(p_1^*M\otimes\mathcal{P}^{-1}) \to p_1^*(L\otimes M)$$

on $X\times T$. The sheaf $p_{2*}(p_1^*(L\otimes M))$ is isomorphic to the trivial bundle $\mathcal{O}_T \times H^0(X, L\otimes M)$. In addition, there exists a nonempty open set $T_0 \subset T$ such that $\mathcal{V}_1$ and $\mathcal{V}_2$ are locally free on T_0. We denote by V_1, V_2 the vector bundles associated to the restriction of $\mathcal{V}_1$, $\mathcal{V}_2$ to T_0. For each $P\in T_0$ there are natural identifications $V_{1,P} \cong H^0(X, L+P)$ and $V_{2,P} \cong H^0(X, M-P)$. In particular we have $\rho_1 \geq a+1$ and $\rho_2 \geq b+1$. Composing the natural bilinear map $V_1 \times_{T_0} V_2 \to V_1\otimes V_2$ with the restriction of ψ, we obtain a morphism $V_1\times_{T_0} V_2 \to T\times H^0(X, L\otimes M)$. In turn, passing to the projectivized bundles and composing with the projection $T_0 \times |L\otimes M| \to |L\otimes M|$, this induces a morphism $\phi\colon \mathbb{P}(V_1)\times_{T_0}\mathbb{P}(V_2) \to |L\otimes M|$ such that the restriction of ϕ at the fibers over $P\in T_0$ corresponds to the natural map of linear systems $|L+P|\times|M-P| \to |L\otimes M|$. The map ϕ has finite fibers, since each element of $|L\otimes M|$ can be decomposed as the sum of two effective divisors in a finite number of ways. It follows that the dimension of $|L\otimes M|$ is greater than or equal to $\dim \mathbb{P}(V_1)\times_{T_0}\mathbb{P}(V_2) = \rho_1+\rho_2+t-2 \geq a+b+t$. $\square$

3. Birationality criteria

In this section we exploit the techniques of 2.2, 2.3 and 2.4 to give criteria for the birationality of maps between varieties of maximal Albanese dimension.

Theorem 3.1. *Let $f\colon X\to Y$ be a generically finite morphism of smooth projective varieties of maximal Albanese dimension. If the induced maps $H^i(Y, \omega_Y\otimes P) \to H^i(X, f_*\omega_X\otimes P)$ are isomorphisms for all $i\geq 0$ and all $P\in \mathrm{Pic}^0(Y)$, then f is birational.*

Proof. Denote by $j\colon \omega_Y \to f_*(\omega_X)$ the natural inclusion. Pushing forward to $\mathrm{Alb}(Y)$, one has an inclusion $\mathrm{alb}_{Y*}(\omega_Y) \hookrightarrow \mathrm{alb}_{Y*}(f_*(\omega_X))$. By Theorem 2.6, c), the sheaves $R^i\mathrm{alb}_{Y*}(f_*\omega_X)$ and $R^i\mathrm{alb}_{Y*}\omega_Y$ vanish for all $i>0$, hence for all $P\in Pic^0(Y)$ the Leray spectral sequence gives natural isomorphisms $H^i(Y,\omega_Y\otimes P)\cong H^i(A, \mathrm{alb}_{Y*}(\omega_Y)\otimes P)$ and $H^i(Y, f_*\omega_X\otimes P) \cong H^i(A, \mathrm{alb}_{Y*}(f_*(\omega_X))\otimes P)$. Thus for all $P\in\mathrm{Pic}^0(Y)$ and $i\geq 0$ we have isomorphisms

$$H^i(A, \mathrm{alb}_{Y*}(\omega_Y)\otimes P) \xrightarrow{\sim} H^i(A, \mathrm{alb}_{Y*}(f_*(\omega_X))\otimes P) \quad .$$

By Corollary 2.3, the sheaves $\mathrm{alb}_{Y*}(\omega_Y)$ and $\mathrm{alb}_{Y*}(f_*(\omega_X))$ are isomorphic and, in particular, they have the same rank at the generic point of $\mathrm{alb}_Y(Y)$. Thus the degree of $X\to Y$ is equal to 1, i.e. f is birational. $\square$

Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties of the same dimension. Let $X \to V$ and $Y \to W$ be birational models of the respective Iitaka fibrations. We have an inclusion of linear series

$$|mK_Y| \xrightarrow{f^*} |mf^*K_Y| \xrightarrow{+mK_{X/Y}} |mK_X|.$$

Since $Y \to W$ is defined by sections of $|mK_Y|$ for m sufficiently big and divisible, and since the pull-backs of these sections correspond sections of $|mK_X|$, one sees that the induced morphism $X \to W$ factors through a rational map $V \to W$. If $P_m(X) = P_m(Y)$ for m sufficiently big and divisible, then $V \to W$ is birational. In particular, if Y is of general type, then X is birational to Y.

It would be interesting to obtain effective versions of this result. In general this seems to be a very complicated problem and it is not clear what bounds to expect. However when Y is of maximal Albanese dimension we obtain some satisfactory results.

Theorem 3.2. *Let $f\colon X \to Y$ be a dominant morphism of smooth n-dimensional projective varieties, Y of maximal Albanese dimension, and let $X \to V$, $Y \to W$ be the Iitaka fibrations of X, respectively Y. If $P_2(X) = P_2(Y)$, then the induced map $V \to W$ has connected fibers. In particular, if Y is of general type, then f is birational.*

Proof. If $\kappa(Y) = 0$, then W is a point and the claim is of course true. Therefore we may assume $\kappa(Y) > 0$.

By [CH3], §1.1, we may assume that V and W are smooth and that the Iitaka fibrations of X and Y fit in a commutative diagram where $A := \mathrm{Alb}(Y)$ and $S := \mathrm{Alb}(W)$

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{a} & A \\ \downarrow & & \downarrow{\scriptstyle p_Y} & & \downarrow \\ V & \longrightarrow & W & \xrightarrow{s} & S. \end{array}$$

Consider the induced maps $p_X\colon X \to W$, $\pi_X\colon X \to S$ and $\pi_Y\colon Y \to S$. The claim will follow if we show that p_X has connected fibers. Fix H ample on S. The map $s\colon W \to S$ is generically finite onto its image by Proposition 2.1, a), hence s^*H is nef and big on W. For $m \gg 0$, the linear system $|mK_Y - \pi_Y^*H|$ is nonempty. Let B be a general member of $|mK_Y - \pi_Y^*H|$ and let $\tilde{B} := f^*B + mK_{X/Y}$. We may assume that both B and $\tilde{B}$ have normal crossings support. Define

$$L_Y := \omega_Y(-\lfloor B/m \rfloor) \equiv \pi_Y^*(H/m) + \{B/m\}$$

and

$$L_X := \omega_X(-\lfloor \tilde{B}/m \rfloor) = f^*\omega_Y(-\lfloor (f^*B)/m \rfloor) \equiv \pi_X^*(H/m) + \{\tilde{B}/m\}.$$

Step 1. $K_{X/Y} - \lfloor (f^*B)/m \rfloor + f^*\lfloor B/m \rfloor$ *is effective.*

Let $B/m = \sum_1^s b_i B_i$, where the B_i are distinct prime divisors. If the b_i are integers, then $\lfloor (f^*B)/m \rfloor = f^*\lfloor B/m \rfloor$. So it is enough to consider the case $0 \leq b_i < 1$, i.e. $\lfloor B/m \rfloor = 0$. Let $P \in Y$ be a point such that $P \in B_i$ for $1 \leq i \leq s$ and let $(y_1, ..., y_n)$ be local coordinates centered in P such that y_i is a local equation for B_i, $i = 1 \dots s$. Let Q be a point such that $f(Q) = P$ and let E be a component of $\tilde{B}$ containing Q. Choose local coordinates $(x_1 \dots x_n)$ around Q such that x_1 is a local equation for E. Assume that E is a component of f^*B_i for $i = 1 \dots t$, so that for $i = 1 \dots t$ we have $f^*y_i = x_1^{n_i}\epsilon_i$, with $n_i > 0$ and ϵ_i a regular function that does not vanish identically on E. A local equation for $K_{X/Y}$ around p is given by the determinant of the Jacobian matrix $(\frac{\partial f^* y_i}{\partial x_j})$, which is easily seen to vanish on E to order at least $(\sum_1^t n_i) - 1$. So the coefficient of E in $K_{X/Y} - \lfloor (f^*B)/m \rfloor + f^*\lfloor B/m \rfloor$ is greater than or equal to $\left(\sum_1^t (n_i - \lfloor n_i b_i \rfloor)\right) - 1 \geq 0$.

Step 2. *There is a map of sheaves* $\omega_Y \otimes L_Y \to f_*(\omega_X \otimes L_X)$ *inducing an isomorphism on global sections.*

Since Y is of Albanese general type, we have $P_m(Y) > 0$ for every $m \geq 1$. After replacing X, Y by appropriate birational models, we may assume that

$$|2K_Y| = F_2 + |M_2|$$

where $|M_2|$ is free and $B + F_2$ has normal crossings, $B \in |mK_Y - \pi_Y^* H|$ being the divisor chosen before.

We wish to define a new divisor $B' \in |m'K_Y - \pi_Y^* H|$ such that $\lfloor B'/m' \rfloor \prec F_2$. To this end, pick $D = F_2 + D' \in |2K_Y|$ such that $D + B$ has normal crossings support and D' is smooth, not contained in the support of B. Let $m' := m + 2l$ and $B' := B + lD \in |m'K_Y - \pi_Y^* H|$. We may write $D = \sum d_i B_i + D'$ and $B = \sum b_i B_i$. For all $l \gg 0$ we have that the multiplicity of B' along B_i satisfies

$$mult_{B_i} \lfloor B'/m' \rfloor = \lfloor \frac{b_i + l d_i}{m + 2l} \rfloor \leq \lfloor \frac{b_i}{2l} + \frac{d_i}{2} \rfloor \leq d_i = mult_{B_i} F_2.$$

We now replace B by B'. It follows that $h^0(\omega_Y \otimes L_Y) = P_2(Y)$.

By Step 1, there is an injection of sheaves $f^*(\omega_Y \otimes L_Y) \to \omega_X \otimes L_X$, corresponding to the effective divisor $K_{X/Y} - \lfloor (f^*B)/m \rfloor + f^*\lfloor B/m \rfloor$. Pushing forward to Y, one gets an inclusion $\omega_Y \otimes L_Y \to f_*(\omega_X \otimes L_X)$. Since $h^0(\omega_Y \otimes L_Y) = P_2(Y) = P_2(X) \geq h^0(f_*(K_X + L_X))$, the corresponding map on global sections is an isomorphism.

Step 3. $(\pi_X)_*(\omega_X \otimes L_X) = (\pi_Y)_*(\omega_Y \otimes L_Y)$.

By step 2, there is an injection of sheaves $\omega_Y \otimes L_Y \to f_*(\omega_X \otimes L_X)$ inducing an isomorphism on global sections. Pushing forward via π_Y we obtain an exact sequence of sheaves on S

$$0 \to \pi_{Y*}(\omega_Y \otimes L_Y) \to \pi_{X*}(\omega_X \otimes L_X) \to \mathcal{Q} \to 0.$$

By Theorem 2.6, b), for all $i > 0$, $P \in \operatorname{Pic}^0(S)$,

$$H^i(S, \pi_{Y*}(\omega_Y \otimes L_Y) \otimes P) = H^i(S, \pi_{X*}(\omega_X \otimes L_X) \otimes P) = 0.$$

Therefore $h^i(\mathcal{Q} \otimes P) = 0$ for all $i > 0$ and $P \in \operatorname{Pic}^0(S)$ and we have $h^0(\mathcal{Q} \otimes P) = \chi(\mathcal{Q} \otimes P) = \chi(\mathcal{Q})$. By Step 2, the map

$$H^0(\pi_{Y*}(\omega_Y \otimes L_Y)) \to H^0(\pi_{X*}(\omega_X \otimes L_X))$$

is an isomorphism. Therefore we have $0 = h^0(\mathcal{Q}) = h^0(\mathcal{Q} \otimes P)$ and hence $\mathcal{Q} = 0$ by Lemma 2.2.

Step 4. *$F_{X/W}$ is connected.*

Since $s \colon W \to S$ is generically finite onto its image, it follows from Step 3 that the generic rank of $p_{X*}(\omega_X \otimes L_X)$ is equal to the generic rank of $p_{Y*}(\omega_Y \otimes L_Y)$. Let $F_{Y/W}$ be a general fiber of $Y \to W$. Recalling that $\omega_Y \otimes L_Y$ is effective and that $\kappa(F_{Y/W}) = 0$, we have $0 < h^0(F_{Y/W}, \omega_Y \otimes L_Y|_{F_{Y/W}}) \leq P_2(F_{Y/W}) \leq 1$. This shows that the generic rank of $p_{Y*}(\omega_Y \otimes L_Y)$ is equal to 1. Since $\omega_X \otimes L_X$ is effective, the generic rank of $p_{X*}(\omega_X \otimes L_X)$ is greater than or equal to the number of connected components of the generic geometric fiber. Therefore, $F_{X/W}$ is connected. $\square$

4. Abelian varieties and theta divisors

In this section we apply the results of §3 to give some new characterizations of abelian varieties and theta divisors. We will need the following result:

Proposition 4.1. *Let $f \colon X \to A$ be a morphism from a smooth projective variety of dimension n to an abelian variety of dimension $n+1$. If $f^* \colon H^0(A, \Omega^n_A) \to H^0(X, \Omega^n_X)$ is an isomorphism, then $f^* \colon H^0(A, \Omega^i_A) \to H^0(X, \Omega^i_X)$ is an isomorphism for $0 \leq i \leq n$.*

Proof. The assumption implies that $\bar{X} := f(X)$ is a divisor of A, hence f is generically finite onto its image. In addition, $\bar{X}$ is of general type, since otherwise, by [Ue, 10.9], $\bar{X}$ would be a pull-back of a divisor from a quotient $\bar{A}$ of A of dimension $d \leq n$. In that case, if $\omega_1, \ldots, \omega_d$ are the pull-backs of the elements of a basis of $H^0(\bar{A}, \Omega^1_{\bar{A}})$, then the restriction of $\omega_1 \wedge \cdots \wedge \omega_d$ to $\bar{X}$ is zero, contradicting the assumptions. Notice that the map $f^* \colon H^0(A, \Omega^i_A) \to H^0(X, \Omega^i_X)$ is injective for every $i \leq n$ by the assumption that it is injective for $i = n$. The proof of Corollary 3.11 of [Mo] gives $h^0(\Omega^i_A) = h^0(\Omega^i_X)$ and thus $H^0(A, \Omega^i_A) \cong H^0(X, \Omega^i_X)$ for $0 \leq i \leq n$. $\square$

The result that follows generalizes Theorem 2.5.

Proposition 4.2. *Let $f \colon X \to A$ be a morphism from a smooth variety of dimension n to an abelian variety of dimension $n+1$. If $h^0(X, \omega_X \otimes f^*P) = 1$ for all $\mathcal{O}_A \neq P \in \operatorname{Pic}^0(A)$ and the map*

$$f^* \colon H^0(A, \Omega^n_A) \to H^0(X, \Omega^n_X)$$

is an isomorphism, then f is birational onto its image $\bar{X}$ and $\bar{X}$ is a principal polarization.

Proof. Let $\tilde{X}$ be a desingularization of $\bar{X}$. We may assume that f factors through $\tilde{f}\colon X \to \tilde{X}$ and $\phi\colon \tilde{X} \to A$.

Step 1. *$\bar{X}$ is a principal polarization.*

Notice first of all that, by the injectivity of

$$f^*\colon H^0(A, \Omega^n_A) \to H^0(X, \Omega^n_X),$$

$\bar{X}$ is a divisor and generates A. In particular, X has maximal Albanese dimension. By Theorem 2.5, to prove that $\bar{X}$ is a principal polarization it is enough to show that:

a) $h^i(\omega_X) = \binom{n+1}{i+1}$ for all $0 \leq i \leq n$;

b) $h^i(\omega_X \otimes f^*P) = 0$ for all $i \geq 1$ and $\mathcal{O}_A \neq P \in \mathrm{Pic}^0(A)$.

Condition a) follows directly from Proposition 4.1.

To prove condition b), assume by way of contradiction that $V := \cup_{i\geq 1} V^i(X, A, \omega_X)$ contains a point $P \neq \mathcal{O}_X$. By Theorem 2.10, a), and Remark 2.11, V is a proper subset of A. Let T be a component of V, let $P \in T$ be a general point and let $v \in H^1(A, \mathcal{O}_A)$ be a vector that is not tangent to T at P. By Theorem 2.10), c) and Remark 2.11, the complex $(H^i(X, \omega_X \otimes P), \wedge v)$ is exact for $i \geq 1$. In particular, we have $\chi(\omega_X \otimes P) \leq h^0(\omega_X \otimes P)$, with equality holding iff the map $H^0(X, \omega_X \otimes P) \stackrel{\wedge v}{\to} H^1(X, \omega_X \otimes P)$ is zero. Since X is of maximal Albanese dimension, it is easy to see that there exists $\sigma \in H^0(A, \Omega^1_A)$ such that the map $H^0(X, \Omega^{n-1}_X \otimes P) \stackrel{\wedge \sigma}{\to} H^0(X, \omega_X \otimes P)$ is non-zero. If we denote by $v \in H^1(A, \mathcal{O}_A)$ the conjugate of σ, then by Theorem 2.10 c) v is not tangent to $V^1(\omega_X)$ at P. By Hodge conjugation and Serre duality with twisted coefficients (cf. [GL1], 2.5), the map $H^0(X, \omega_X \otimes P) \stackrel{\wedge v}{\to} H^1(X, \omega_X \otimes P)$ is non-zero. Therefore, $\chi(\omega_X \otimes P) < h^0(\omega_X \otimes P) = 1$. On the other hand, $\chi(\omega_X \otimes P) = \chi(\omega_X)$, and $\chi(\omega_X) = 1$ as observed above. Thus we have a contradiction and the proof of Step 1 is complete.

Step 2. *The map f is birational.*

Since f factors through $\phi\colon \tilde{X} \to A$, it is clear that $\phi^*\colon H^0(A, \Omega^n_A) \to H^0(\tilde{X}, \Omega^n_{\tilde{X}})$ is an isomorphism. Then Proposition 4.1 implies that the maps $\phi^*\colon H^0(A, \Omega^i_A) \to H^0(\tilde{X}, \Omega^i_{\tilde{X}})$ are isomorphisms for $0 \leq i \leq n$. In particular one has $\chi(\omega_{\tilde{X}}) = 1$. In addition, for every P the map $\tilde{f}^*\colon H^0(\tilde{X}, \omega_{\tilde{X}} \otimes P) \to H^0(X, \omega_X \otimes f^*P)$ is injective and we have $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes P) \leq h^0(X, \omega_X \otimes f^*P) = 1$. Now the same argument as in Step 1 can be used to show that $h^i(\tilde{X}, \omega_{\tilde{X}} \otimes P) = 0$ for $P \neq \mathcal{O}_A$ and $i > 0$. Thus for every $P \neq \mathcal{O}_A$ we have $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes P) = \chi(\omega_{\tilde{X}} \otimes P) = \chi(\omega_{\tilde{X}}) = 1$. It follows that for $P \neq \mathcal{O}_A$ the map $H^0(\tilde{X}, \omega_{\tilde{X}} \otimes P) \to H^0(\tilde{X}, \tilde{f}_*\omega_X \otimes P) = H^0(X, \omega_X \otimes f^*P)$ is an isomorphism, being a non

trivial map of $1-$dimensional vector spaces. So the assumption of Theorem 3.1 is satisfied for $P \neq \mathcal{O}_A$. For $P = \mathcal{O}_A$ it follows from the fact that $H^0(\tilde{X}, \Omega^i_{\tilde{X}}) \to H^0(X, \Omega^i_X)$ is an isomorphism by using Hodge conjugation and Serre duality. Thus f has degree 1 by Theorem 3.1. □

Corollary 4.3. *Let X be a smooth projective variety such that*

$$P_1(X) = q(X) = \dim(X) + 1.$$

If X is of maximal Albanese dimension and $\mathrm{Alb}(X)$ *is simple, then X is birational to a theta divisor in* $\mathrm{Alb}(X)$.

Proof. We write $n = \dim(X)$, and $A := \mathrm{Alb}(X)$. Consider the map $H^0(A, \Omega^n_A) \to H^0(X, \Omega^n_X)$. If it is not an isomorphism, then there exist $1-$forms $\omega_1, \dots, \omega_n \in H^0(A, \Omega^1_A) = H^0(X, \Omega^1_X)$ such that $\omega_1 \wedge \dots \wedge \omega_n = 0$ on X. By the Generalized Castelnuovo - de Franchis Theorem (cf. for instance [Ca], Theorem 1.14) it follows that there exists a fibration $p: X \to Y$ $(\dim(Y) < \dim(X))$ such that $\omega_1 \dots \omega_n$ are pull-backs from Y. Notice that $n \leq q(Y) < q(X)$, since X has maximal Albanese dimension. Then the induced morphism $A \to \mathrm{Alb}(Y)$ is surjective and is not an isogeny, contradicting the assumption that A be simple. Thus $H^0(A, \Omega^n_A) \to H^0(X, \Omega^n_X)$ is an isomorphism and, by Proposition 4.1, $H^0(A, \Omega^i_A) \to H^0(X, \Omega^i_X)$ is also an isomorphism for all $i \leq n$. In particular, we have $\chi(\omega_X) = 1$. So the claim will follow from Proposition 4.2 if we show that $h^i(\omega_X \otimes P) = 0$ for all $i > 0$ and $P \neq \mathcal{O}_X$.

By the generic vanishing theorems (Theorem 2.10), the components of $V^i(\omega_X)$ are torsion translates of subtori of $\mathrm{Pic}^0(X)$ of codimension at least i. Since A is simple, it follows that for all $i > 0$ the sets $V^i(\omega_X)$ consist of finitely many torsion points in $P \in \mathrm{Pic}^0(X)$. Since $V^i(\omega_X) \subset V^1(\omega_X)$ for $i \geq 1$ by Theorem 2.10, d), we have $V^i(\omega_X) = \{\mathcal{O}_X\}$ for $i > 0$ by Proposition 2.12. □

When $\mathrm{Alb}(X)$ is simple, it is also possible, using a result of Ein and Lazarsfeld, to characterize abelian varieties in terms of $P_1(X)$.

Lemma 4.4. *If X is a smooth projective variety such that $P_1(X) > 0$, $q(X) = \dim(X)$ and $V^0(\omega_X)$ has dimension 0, then X is birational to* $\mathrm{Alb}(X)$.

Proof. By [EL1, Prop. 2.2], $X \to \mathrm{Alb}(X)$ is surjective, hence X has maximal Albanese dimension.

By an argument due to Ein and Lazarsfeld (cf. [CH1, Theorem 1.7]), X is birational to $\mathrm{Alb}(X)$. □

Corollary 4.5. *Let X be a nonsingular projective variety of dimension n. If X is of maximal Albanese dimension,* $\mathrm{Alb}(X)$ *is simple, $P_1(X) = 1$ and $q(X) = n$, then X is birational to* $\mathrm{Alb}(X)$.

Proof. By assumption the Albanese map $\mathrm{alb}_X \colon X \to \mathrm{Alb}(X)$ is generically finite and surjective. By a result of Ueno (cf. [Mo, 3.4]) one has $h^{i,0}(X) = \binom{n}{i}$ for all $i \in [1, ..., n]$. By Hodge symmetry and Serre duality, one has that $h^{n-i}(\omega_X) = \binom{n}{i}$. In particular $\chi(\omega_X) = 0$. Since A is simple, $V^i(\omega_X)$ is $0-$dimensional for $i > 0$. Since $\chi(\omega_X) = 0$, then $V^0(\omega_X)$ is also $0-$dimensional and the claim follows from Lemma 4.4. □

5. The Albanese map

Here we give a strengthening of a result of Kollàr on the surjectivity of the Albanese map.

Theorem 5.1. *Let X be a smooth projective variety. If $P_2(X) = 1$ or $0 < P_m(X) \leq 2m - 3$ for some $m \geq 3$, then $\mathrm{alb}_X \colon X \to \mathrm{Alb}(X)$ is surjective.*

Proof. If $P_2(X) = 1$ then this is [CH3, Theorem 1]. If $P_3(X) = 1$ or $0 < P_m(X) \leq 2m - 6$ for some $m \geq 4$, this is [Ko4, Theorem 11.2]. We will follow the proof of [Ko4, Theorem 11.2]. Assume that $\mathrm{alb}_X \colon X \to \mathrm{Alb}(X)$ is not surjective. By [Ue, 10.9], up to replacing X by an appropriate birational model, there is a morphism $f \colon X \to Z$ where Z is a smooth variety of general type of dimension ≥ 1, such that the Albanese map $a \colon Z \to S := \mathrm{Alb}(Z)$ is birational onto its image. We denote by $F_{X/Z}$ a general fiber of f. In particular, notice that, $\dim(S) \geq 2$.

Step 1. *There exists an ample divisor L on Z such that, after replacing X by an appropriate birational model, there exist an integer $r \gg 0$ and a divisor $B \in |rm(K_X + (m-2)K_{X/Z}) - f^*L)|$ such that B has normal crossings support and*

$$\lfloor \frac{B|_{F_{X/Z}}}{rm} \rfloor \prec F$$

where $|mK_{F_{X/Z}}| = |H| + F$.

We remark first of all that the linear system $|mK_{F_{X/Z}}|$ is nonempty, since $P_m(X) > 0$. Let L_0 be an ample $\mathbb{Q}-$divisor on Z such that $K_Z - 2L_0$ is big. By the proof of [Ko4, 10.2], after replacing X by an appropriate birational model, for r sufficiently big and divisible there exists a divisor $D \in |rm(m-1)K_{X/Z} + rmf^*L_0|$ such that the restriction of D to the general fiber $F_{X/Z}$ is equal to $\bar{H} + r(m-1)F$, where $|mK_{F_{X/Z}}| = |H| + F$ and $\bar{H} \in |r(m-1)H|$ is a general smooth member. Let G be a general member of $|rm(K_Z - 2L_0)|$ and set $L = rmL_0$. Let $B = f^*G + D \in |rm(K_X + (m-2)K_{X/Z} - f^*L)|$. After replacing X by an appropriate birational model, we may assume that B has normal crossings support. Then $B|_{F_{X/Z}} = D|_{F_{X/Z}} = (\bar{H} + r(m-1)F)$. Therefore

$$\lfloor \frac{B|_{F_{X/Z}}}{rm} \rfloor = \lfloor \frac{\bar{H}}{rm} + \frac{m-1}{m} F \rfloor \prec F.$$

Step 2. $\dim|(2K_X+(m-2)K_{X/Z})\otimes f^*P|\geq 1$ *for all* $P\in \mathrm{Pic}^0(Z)$.

Let B be a divisor as in Step 1. Define

$$M := K_X + (m-2)K_{X/Z} - \lfloor \frac{B}{rm} \rfloor \equiv \frac{f^*L}{mr} + \{\frac{B}{rm}\}.$$

One has:

$$H^0(F_{X/Z}, \mathcal{O}_{F_{X/Z}}(K_X+M)) = H^0(F_{X/Z}, \mathcal{O}_{F_{X/Z}}(mK_{F_{X/Z}} - \lfloor \frac{B}{rm} \rfloor))$$

$$= H^0(F_{X/Z}, mK_{F_{X/Z}}) > 0.$$

In particular $f_*(K_X+M)\neq 0$, and hence, by Theorem 2.8, $h^0(f_*(K_X+M)\otimes P)$ is a nonzero constant, independent of $P\in \mathrm{Pic}^0(Z)$.

We are going to show that this constant is >1. Indeed assume that it is equal to 1 and consider the Albanese map $s\colon Z\to S:=\mathrm{Alb}(Z)$. By Lemma 2.7 we have $h^i(S, s_*f_*(K_X+M)) = h^i(Z, f_*(K_X+M))$. If $i>0$, then $h^i(Z, f_*(K_X+M))=0$ by Theorem 2.6, b), and by Proposition 2.4 it follows that $s_*f_*(K_X+M)$ is supported on an abelian subvariety of S. However, by Theorem 2.6, a), $f_*(K_X+M)$ is a torsion free sheaf, and thus its support is Z. Since s is birational, the support of $s_*f_*(K_X+M)$ is $s(Z)$, contradicting the fact Z is of general type. This shows that for every $P\in \mathrm{Pic}^0(Z)$ we have

$$h^0((2K_X+(m-2)K_{X/Z})\otimes P) \geq h^0(f_*(K_X+M)\otimes P)\geq 2.$$

Step 3.

We have (cf. [Ko4, 11.2]) that $P_1(Z)\geq \dim(Z)+1$, $P_r(Z)\geq 2r-1$ for all $r\geq 2$, and if $\dim(Z)\geq 2$ then $P_r(Z)\geq 2r$. It follows that by Lemma 2.15, $h^0(rK_Z+P)\geq 2r-1$ for all $r\geq 2$ and $P\in \mathrm{Pic}^0(Z)$, and if $\dim(Z)\geq 2$ then $h^0(rK_Z+P)\geq 2r$.

We now apply Lemma 2.17 with $T:=f^*\mathrm{Pic}^0(Z)$, $L=2K_X+(m-2)K_{X/Z}$, $M=(m-2)f^*K_Z$, so that $L\otimes M = mK_X$. Recalling that $\dim \mathrm{Pic}^0(Z)\geq 2$, for $m\geq 4$ we get $\dim|mK_X|\geq 2(m-2)-2+1+2 = 2m-3$, the required contradiction. For $m=3$ and $\dim(Z)=1$, we have $h^0(K_Z-P)\geq 1$ for every $P\in \mathrm{Pic}^0(Z)$ and Lemma 2.17 gives again $\dim|3K_X|\geq 3 = 2m-3$. Finally, for $m=3$ and $\dim(Z)\geq 2$, we have $h^0(K_Z)\geq 3$ and the contradiction follows by considering the morphism of linear series

$$|2K_X+K_{X/Z}|\times f^*|K_Z|\to |3K_X|.$$

$\square$

It is natural to ask whether these bounds are optimal. For example, does there exists an X with $P_3(X)=4$, such that alb_X is not surjective?

6. Varieties with $P_3(X) = 2$ and $q(X) = \dim(X)$

In this section we give an explicit birational description of the varieties X with $P_3(X) = 2$ and $q(X) = \dim(X)$. A key step of the proof is the use of Theorem 5.1. The precise statement of the result is as follows:

Theorem 6.1. *Let X be a smooth projective variety. Then $P_3(X) = 2$ and $q(X) = \dim(X)$ iff:*

a) *there is a surjective map $\phi\colon \mathrm{Alb}(X) \to E$, where E is a curve of genus 1;*
b) *$\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$ is birational to a smooth double cover of $\mathrm{Alb}(X)$ defined by $Spec(\mathcal{O}_{\mathrm{Alb}(X)} \oplus P \otimes \phi^*L^{-1})$, where L is a line bundle of E of degree 1 and $P \in \mathrm{Pic}^0(X) \setminus \phi^*\mathrm{Pic}^0(E)$ is 2-torsion. The branch locus of the double cover is the union of two distinct fibers of ϕ.*

Proof. If X is the double cover described in the statement, then the standard formulas for double covers give $P_1(X) = 1$ and, for $m \geq 2$, $P_m(X) = m$ if m is even and $P_m(X) = m - 1$ if m is odd.

Assume now that $P_3(X) = 2$ and $q(X) = \dim(X)$. By Theorem 5.1, $X \to \mathrm{Alb}(X)$ is surjective and thus X has maximal Albanese dimension. Let $f\colon X \to Y$ denote the Iitaka fibration of X. We have a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{alb}_X} & \mathrm{Alb}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} \\ Y & \xrightarrow{\mathrm{alb}_Y} & \mathrm{Alb}(Y). \end{array}$$

Moreover, by Proposition 2.1, $K := ker f_*$ is connected and there exists an abelian variety P isogenous to K and birational to $F := F_{X/Y}$. Let

$$G := ker\left(\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(F)\right).$$

Then G is the union of finitely many translates of $\mathrm{Pic}^0(Y)$ corresponding to the finite group

$$G/\mathrm{Pic}^0(Y) \cong ker\left(\mathrm{Pic}^0(K) \to \mathrm{Pic}^0(F)\right).$$

For any line bundle L on X, define (cf. §2)

$$V^i(L) := \{P \in \mathrm{Pic}^0(X) | h^i(L \otimes P) \neq 0\}.$$

Since $(\omega_X^m \otimes P)|F = \omega_F^m \otimes P$, it follows that $V^0(\omega_X^m) \subseteq G$ for every $m \geq 0$. By [CH2] Lemma 2.2, one sees that for all $P \in G \setminus \mathrm{Pic}^0(Y)$, the dimension of $V^0(\omega_X) \cap (P + \mathrm{Pic}^0(Y))$ is ≥ 1. Moreover, by Corollary 2.14 the only possible 0-dimensional component of $V^0(\omega_X)$ is the origin. The proof is divided into several steps.

Step 1. *$V^0(\omega_X^2) = G$.*

Since $h^0(\omega_X) > 0$, we have $V^0(\omega_X) \subset V^0(\omega_X^2)$. By Proposition 2.15, we have $h^0(2K_X + P) = P_2(X) > 0$ for every $P \in \mathrm{Pic}^0(Y)$, namely $V^0(\omega_X^2) \cap \mathrm{Pic}^0(Y) = \mathrm{Pic}^0(Y)$. Consider an irreducible component $S = Q + \mathrm{Pic}^0(Y)$ of G, where $Q \in \mathrm{Pic}^0(X) \setminus \mathrm{Pic}^0(Y)$ is torsion. Then by [CH2, Lemma 2.2] $V^0(\omega_X) \cap S$ has positive dimension, in particular it is nonempty. Thus $V^0(\omega_X^2) \cap S$ is also nonempty and Proposition 2.15 implies again $V^0(\omega_X^2) \cap S = S$.

Step 2. *If T is an irreducible component of $V^0(\omega_X)$, then* $\dim(T) \leq 1$.
Since $T \subset G$, we have also $-T \subset G$, and hence by Step 1, $h^0(2K_X - Q) > 0$ for all $Q \in T$. Applying Lemma 2.17 with $L = K_X$ and $M = 2K_X$ we get $\dim(T) \leq \dim |3K_X| = 1$.

Step 3. *For every $P \in V^0(\omega_X)$ one has $h^0(K_X + P) = 1$.*
Assume that there is $P \in V^0(\omega_X)$ such that $h^0(K_X + P) \geq 2$. Then we have $h^0(2K_X + 2P) \geq 3$ and thus, by Proposition 2.15, we have $h^0(2K_X + 2P + R) \geq 3$ for all $R \in \mathrm{Pic}^0(Y)$. Since $P \in G$, then $-2P \in G$. Let P_0 be a point of $V^0(\omega_X) \cap -2P + \mathrm{Pic}^0(Y)$ (such a point exists by [CH2, Lemma 2.2]). Then from $h^0(2K_X - P_0) \geq 3$ and $h^0(K_X + P_0) > 0$ it follows immediately $h^0(3K_X) \geq 3$, against the assumptions.

Step 4. *For any $P \in G \setminus \mathrm{Pic}^0(Y)$, $h^0(2K_X + P) \leq 1$.*
Let $P \in G \setminus \mathrm{Pic}^0(Y)$ be such that $h^0(2K_X + P) \geq 2$. By Proposition 2.15, one has $h^0(2K_X + R + P) \geq 2$ for all $R \in \mathrm{Pic}^0(Y)$. Since $P \in G \setminus \mathrm{Pic}^0(Y)$ also $-P \in G \setminus \mathrm{Pic}^0(Y)$. Let T be an irreducible component of $V^0(\omega_X) \cap -P + \mathrm{Pic}^0(Y)$. Then by Step 1 and Proposition 2.15, one has $h^0(2K_X - Q) \geq 2$ for all $Q \in T$. Since T has positive dimension by [CH2, Lemma 2.2], Lemma 2.17 gives $\dim |3K_X| \geq 2$, a contradiction.

Step 5. $V^0(\omega_X) \cap \mathrm{Pic}^0(Y) = \{\mathcal{O}_X\}$.
Recall that by Corollary 2.14 $\{\mathcal{O}_X\}$ is the only possible $0-$dimensional component of $V^0(\omega_X)$. Let T be a positive dimensional component of $V^0(\omega_X) \cap \mathrm{Pic}^0(Y)$. Then by Theorem 2.10, b), $T = Q + T_0$, where $Q \in \mathrm{Pic}^\tau(Y)$ and T_0 is an abelian subvariety of $\mathrm{Pic}^0(Y)$. For every $P \in T_0$, we have $h^0(K_X + Q + P) > 0$ and $h^0(K_X + Q - P) > 0$. Thus Lemma 2.17 gives $h^0(2K_X + 2Q) \geq 2$. By Proposition 2.15 we have $h^0(2K_X + P) \geq 2$ for every $P \in \mathrm{Pic}^0(Y)$. In particular, we have $h^0(2K_X - P) \geq 2$ and $h^0(K_X + P) \geq 1$ for all $P \in T$. Applying Lemma 2.17 again gives $\dim |3K_X| \geq 2$, a contradiction.

Step 6. $Y \to \mathrm{Alb}(Y)$ *is birational.*
Since X is of Albanese general type, the same is true for Y by Proposition 2.1. So if $\kappa(Y) = 0$ then the claim is true by [Ka]. Assume $\kappa(Y) > 0$. By the commutativity of the diagram at the beginning of the proof, the map $Y \to \mathrm{Alb}(Y)$ is surjective, since $X \to \mathrm{Alb}(X)$ is surjective. By a result of Ein and Lazarsfeld (see also [CH2, Theorem

2.3]) there is an irreducible component $T \subset V^0(Y, \omega_Y)$ of positive dimension. The divisor $K_{X/Y}$ is effective since X has maximal Albanese dimension, hence $\dim\big(V^0(X, \omega_X) \cap \mathrm{Pic}^0(Y)\big) \geq 1$, contradicting Step 5.

Denote by $\pi \colon X \to \mathrm{Alb}(Y)$ the composition of the Albanese map $X \to \mathrm{Alb}(X)$ and of $f_* \colon \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$.

Step 7. *For any $P \in G \setminus \mathrm{Pic}^0(Y)$, there exists a principal polarization N on $\mathrm{Alb}(Y)$ and an effective divisor D on X such that*

$$|2K_X + P| = \pi^*|N| + D.$$

See Step 3 in the proof of [CH1, Theorem 2.4].

Let $T \subset \mathrm{Pic}^0(X)$ be any $1-$dimensional irreducible component of $V^0(\omega_X)$. We recall that by Theorem 2.10, b), T is a translate of an abelian subvariety $\bar{T}$. Let $\mathrm{Alb}(X) \to E := \mathrm{Pic}^0(\bar{T})$ be the dual map of abelian varieties and $\pi_E \colon X \to E$ the induced morphism. By Lemma 2.16, there exists a divisor $D \prec R := Ram(\mathrm{alb}_X) = K_X$, vertical with respect to π_E, such that for all $P \in T$, $F := R - D$ is a fixed divisor of each of the linear series $|K_X + P|$. In other words, we have for all $P \in T$

$$|K_X + P| = F + |V_P|.$$

The divisors in $|V_P|$ are vertical with respect to π_E. To see this, notice that $(K_X - F)|_{F_{X/E}} = D|_{F_{X/E}} = \mathcal{O}_{F_{X/E}}$ so $(K_X - F + P)|_{F_{X/E}} = (V_P)|_{F_{X/E}} = \mathcal{O}_{F_{X/E}}$.

Since $\bar{T} \subset \mathrm{Pic}^0(Y) \subset \mathrm{Pic}^0(X)$, the map π_E factors through the map $\mathrm{Alb}(Y) \to E$, which has connected fibers since it is dual to an inclusion of tori. The map $X \to \mathrm{Alb}(Y)$ has connected fibers by Step 6, hence π_E has connected fibers.

Step 8. *For general $P \in T$, there exists a line bundle L_P of degree* 1 *on E such that $\pi_E^* L_P \prec K_X + P$.*

The divisors V_P are nonempty and vary with $P \in T$. Since V_P is vertical, for $P \in T$ general it contains a smooth fiber of $\pi(e)$, where $e \in E$. So we may set $L_P := \mathcal{O}_E(e)$.

Step 9. *$\kappa(X) = 1$.*

Let T be a component of $V^0(\omega_X)$ of positive dimension. By Step 2, T has dimension 1. As before, we let $E = \mathrm{Pic}^0(\bar{T})$ and we denote by $\pi_E \colon X \to E$ the corresponding map. Let $P \in T$ be a general point. By Step 8, there is a line bundle L_P of degree 1 on E such that $\pi_E^* L_P \prec K_X + P$, and by Step 7 there is a principal polarization N on $\mathrm{Alb}(Y)$ such that $\pi^* N \prec 2K_X - P$. We denote by L the pull-back of L_P to $\mathrm{Alb}(Y)$. There is an inclusion $\pi^*|L \otimes N| \to |3K_X|$, and thus the dimension of $|L \otimes N|$ is ≤ 1. On the other hand, for every $Q \in \bar{T} = \mathrm{Pic}^0(E) \subset \mathrm{Pic}^0(Y)$ we have $h^0(\mathrm{Alb}(Y), L + Q) = 1$ and

$h^0(\mathrm{Alb}(Y), N-Q) = 1$. So Lemma 2.17 gives $\dim |L \otimes N| \geq 1$. It follows that $\dim |L \otimes N| = 1$, and the proof of Lemma 2.17 shows that for every divisor D in $|L \otimes N|$ there is $Q \in \mathrm{Pic}^0(E)$ such that D can be written as $D'_Q + D''_Q$, where $D'_Q \in |L+Q|$ and $D''_Q \in |N-Q|$. Let $|M|$ be the moving part of $|L \otimes N|$. The line bundle M is positive semidefinite, so that there exist a quotient $\bar{A}$ of $\mathrm{Alb}(Y)$ and an ample line bundle $\bar{M}$ on $\bar{A}$ such that M is the pull-back of $\bar{M}$ to $\mathrm{Alb}(Y)$ and $h^0(M) = h^0(\bar{M})$ (cf. [LB], §3.3). Notice that since the continuous system $\{D'_Q\}$ is base point free, every divisor of $|M|$ can be written as $D'_Q + R_Q$ for suitable $Q \in \mathrm{Pic}^0(E)$. Moreover, $R_Q > 0$, since $0 = \dim |L_Q| < 1 = \dim |M|$. So the general divisor of $|M|$ is reducible, and the same is true for the general divisor of $|\bar{M}|$ on $\bar{A}$. Since $\bar{M}$ is ample and has no fixed part, by Theorem 4.5 of [LB] this can only happen if $\bar{A}$ has dimension 1. Since $L \prec M$, it follows that $\bar{A} = E$, namely the moving part of $|L \otimes N|$, and thus also of $|3K_X|$, is a pull-back from E. This condition determines the map $\mathrm{Alb}(Y) \to E$ uniquely and, taking duals, it determines uniquely $\bar{T} \subset \mathrm{Pic}^0(Y)$. So we have shown that all the positive dimensional components of $V^0(\omega_X)$ are translate of the same abelian subvariety $\bar{T} \subset \mathrm{Pic}^0(Y)$ of dimension 1. Since by [CH2, Theorem 2.3], $\mathrm{Pic}^0(Y)$ is generated by the sum of these translates, it follows that $\mathrm{Alb}(Y)$ has dimension 1, namely $\kappa(X) = 1$.

Step 10. $V^0(\omega_X) = G \setminus \mathrm{Pic}^0(Y) \cup \{\mathcal{O}_X\}$.

By Step 6 and Step 9, Y is a smooth curve of genus 1. As we have remarked at the beginning of the proof, $V^0(\omega_X)$ intersects each component of $G \setminus \mathrm{Pic}^0(Y)$ in a set of positive dimension. This shows that $V^0(\omega_X)$ contains $G \setminus \mathrm{Pic}^0(Y)$. The statement now follows from Step 5.

Step 11. alb_X *is of degree 2.*

Since $X \to \mathrm{Alb}(Y)$ and $\mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ have connected fibers, then

$$d = \deg\left(X \to \mathrm{Alb}(X)\right) = \deg\left(F_{X/Y} \to K\right)$$

is just the cardinality of $G/\mathrm{Pic}^0(Y)$. If $d \geq 3$, then there exist elements $P_1, P_2, P_3 \in V^0(\omega_X) \setminus \mathrm{Pic}^0(Y)$ such that $P_1 + P_2 + P_3 = \mathcal{O}_X$. Since $h^0(K_X + P_i + Q) = 1$ for all $Q \in \mathrm{Pic}^0(Y)$ by Step 10, it follows applying Lemma 2.17 that $h^0(2K_X + P_1 + P_2 + Q) \geq 2$ for all $Q \in \mathrm{Pic}^0(Y)$ and similarly that $h^0(3K_X) = h^0(3K_X + P_1 + P_2 + P_3) \geq 3$, a contradiction.

Step 12. $\mathrm{alb}_{X*}\omega_X = \mathcal{O}_A \oplus (f_*)^* L \otimes P$ *where* L *is a principal polarization on* $\mathrm{Alb}(Y)$ *and* $P^{\otimes 2} = \mathcal{O}_X$ *but* $P \notin \mathrm{Pic}^0(Y)$.

By Step 10 and Step 11, we have

$$V^0(\omega_X) = \{\mathcal{O}_X\} \cup (P + \mathrm{Pic}^0(Y))$$

for an appropriate 2-torsion element $P \in \mathrm{Pic}^0(X) \setminus \mathrm{Pic}^0(Y)$. The sheaf $L := f_*(\omega_X \otimes P)$ is torsion free by Theorem 2.6, a) and it has rank 1

since the general fiber $F_{X/Y}$ of f is birational to an abelian variety and the restriction of P to $F_{X/Y}$ is trivial. Since Y is a curve, L is actually a line bundle. In addition, we have $h^0(L\otimes Q) = 1$ for all $Q \in \mathrm{Pic}^0(Y)$. Therefore L is a principal polarization. Similarly the sheaf $M := f_*(\omega_X)$ is a line bundle such that $h^0(M) = 1$ and $h^0(M\otimes Q) = 0$ for all $Q \in \mathrm{Pic}^0(Y) \setminus \{\mathcal{O}_Y\}$, therefore $M = \mathcal{O}_Y$. Let $f_*\colon \mathrm{Alb}(X) \to \mathrm{Alb}(Y) = Y$ and let $\tilde{L} := (f_*)^*L$. There are inclusions of sheaves

$$\mathcal{O}_{\mathrm{Alb}(X)} = (f_*)^*(f_*)_*(\mathrm{alb}_{X*}\omega_X) \to \mathrm{alb}_{X*}\omega_X,$$
$$\tilde{L} = (f_*)^*(f_*)_*(\mathrm{alb}_{X*}\omega_X \otimes P) \to \mathrm{alb}_{X*}(\omega_X \otimes P).$$

There is a corresponding map

$$\psi\colon \mathcal{O}_{\mathrm{Alb}(X)} \oplus \tilde{L}\otimes P \to \mathrm{alb}_{X*}\omega_X.$$

Restricting to a generic fiber K of $\mathrm{Alb}(X) \to \mathrm{Alb}(Y) = Y$, one has that the above map is given by

$$\mathcal{O}_K \oplus P|K \to (\mathrm{alb}_{X*}\omega_X)|K.$$

Let F be a general fiber of $X \to Y$, $a := \mathrm{alb}_X|_F\colon F \to K$. The above map of sheaves is equivalent to the isomorphism

$$\mathcal{O}_K \oplus P|K \cong a_*\omega_F.$$

It follows that ψ is an inclusion of sheaves.

We wish to show that ψ is an isomorphism. To this end, by Corollary 2.2, it suffices to show that ψ induces isomorphisms of cohomology groups

$$H^i\left((\mathcal{O}_{\mathrm{Alb}(X)} \oplus \tilde{L} \otimes P) \otimes Q\right) \to H^i\left(\mathrm{alb}_{X*}(\omega_X)\otimes Q\right)$$

for all $i \geq 0$ and $Q \in \mathrm{Pic}^0(X)$.

For $i = 0$ the map above is injective and it is enough to check that the two vector spaces have the same dimension. By Step 3 and by the description of $V^0(\omega_X)$ that we have given, it follows that for $i = 0$ and any $Q \in \mathrm{Pic}^0(X)$, we have isomorphisms in cohomology. If $i \geq 1$ and Q is not in $\{\mathcal{O}_X\} \cup (\mathrm{Pic}^0(Y) + P)$, then both vector spaces are 0 (cf. Theorem 2.10, d)). We will prove that the above isomorphism holds for all $Q \in P + \mathrm{Pic}^0(E)$. (The proof for $Q = \mathcal{O}_X$ proceeds analogously but is easier). For such a choice of Q one has:

$$H^i\left(\tilde{L}\otimes P \otimes Q\right) = H^i\left((\mathcal{O}_A \oplus \tilde{L}\otimes P) \otimes Q\right).$$

We observe that $\chi(\omega_X) = 0$ by Theorem 2.10, d), since X has maximal Albanese dimension and we have seen that $V^0(\omega_X)$ is a proper subset of $\mathrm{Pic}^0(X)$. Let $W \subset H^1(\mathcal{O}_X)$ be a subspace complementary to the tangent space to $\mathrm{Pic}^0(Y)$. The assumptions of Proposition 2.13 are satisfied for all $Q \in P + \mathrm{Pic}^0(E)$, hence for all $0 \leq j \leq \dim(X)$ and $Q \in P + \mathrm{Pic}^0(E)$ there are isomorphisms

$$H^j(\omega_X\otimes Q) \cong H^0(\omega_X\otimes Q)\otimes \wedge^j W.$$

induced by cup product. The required isomorphism is given by the following commutative diagram

$$\begin{array}{ccc} H^0(\tilde{L}\otimes P\otimes Q)\otimes \wedge^i W & \xrightarrow{\cong} & H^0(\mathrm{alb}_{X*}\omega_X\otimes Q)\otimes \wedge^i W \\ \cong\Big\downarrow & & \Big\downarrow\cong \\ H^i(\tilde{L}\otimes P\otimes Q) & \longrightarrow & H^i(\mathrm{alb}_*\omega_X\otimes Q) \end{array}$$

Step 13. *Conclusion of the proof.*

If $X \xrightarrow{\epsilon} Z \xrightarrow{g} \mathrm{Alb}(X)$ is the Stein factorization of alb_X, then by Proposition 2.9, g is flat and $g_*\mathcal{O}_Z = \mathcal{O}_{\mathrm{Alb}(X)} \oplus \tilde{L}^{-1}\otimes P$. The branch locus of g is reduced, since Z is normal, and it is linearly equivalent to $2\tilde{L}$, thus it consists of two fibers of f_*. □

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